A Re-look at 2nd Order Linear Differential Equation

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Long ago, I acquired the knowledge that for 1st order linear differential equation we can apply the idea of using the auxiliary equation, then form the complementary function and particular integral (PI) and hence state the general solution. We shall call this method the traditional method. However, the method involving integrating factor can find the general solution beyond 1st order linear differential equation. Hence, I was motivated to investigate how we can solve the 2nd order linear differential equation by using the idea of integrating factor.

To avoid complicated manipulations, I started using the idea of integrating factor to find the general solution for the 2nd order linear equation $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = x$. After successful attempt, I crosschecked the answer obtained with the traditional method. Then I went on to generalize the method as follows:

Consider $\dfrac{d^2y}{dx^2} - (\alpha+\beta)\dfrac{dy}{dx} + \alpha\beta y = f(x)$, where α and β are real numbers.

We can rearrange the second order linear equations as:

$$\left[\frac{d^2y}{dx^2} - \alpha\,\frac{dy}{dx}\right] - \beta\,\left[\frac{dy}{dx} - \alpha y\right] = f(x).$$

Applying the idea of integrating factor, multiply both sides of the equation by $e^{-\alpha x}$,

$$\begin{aligned} e^{-\alpha x}\left[\frac{d^2y}{dx^2} - \alpha\,\frac{dy}{dx}\right] - \beta e^{-\alpha x}\,\left[\frac{dy}{dx} - \alpha y\right] &= e^{-\alpha x} f(x) \\ \frac{d}{dx}\,\left[e^{-\alpha x}\,\frac{dy}{dx}\right] - \beta\,\frac{d}{dx}\,[e^{-\alpha x} y] &= e^{-\alpha x} f(x) \\ \frac{d}{dx}\,\left[e^{-\alpha x}\,\left(\frac{dy}{dx} - \beta y\right)\right] &= e^{-\alpha x} f(x). \end{aligned}$$

Let $F(x) = \int^x e^{-\alpha s} f(s)\,ds$, we get:

$$\begin{aligned} e^{-\alpha x}\left(\frac{dy}{dx} - \beta y\right) &= F(x) + c \\ \left(\frac{dy}{dx} - \beta y\right) &= e^{\alpha x}[F(x) + c]. \end{aligned}$$

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Applying the idea of integrating factor, multiply both sides of the equation by $e^{-\beta x}$,

$$\begin{aligned} e^{-\beta x}\left(\frac{dy}{dx}-\beta y\right) &= e^{-\beta x}[e^{\alpha x}(F(x)+c)] \\ \frac{d}{dx}\left(e^{-\beta x}y\right) &= e^{-\beta x}[e^{\alpha x}(F(x)+c)] \\ e^{-\beta x}y &= \int e^{-\beta x}\,[e^{\alpha x}(F(x)+c)]dx \\ y &= e^{\beta x}\int e^{-\beta x}[e^{\alpha x}(F(x)+c)]dx. \end{aligned}$$

This method may seem tedious but it will be able to offer students an alternative method to understand why there is a need to multiply "x" to the Particular Integral (PI) if the PI is repeating the complementary function. In fact this method also prove that in the event that the auxiliary equation has repeated roots, say δ, then the complementary function is $y = (Ax + B)e^{\delta x}$.

Example to illustrate the above concept:

Consider the following differential equation:

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = x.$$

We can either consider $\alpha = 0$ and $\beta = 1$ or $\alpha = 1$ and $\beta = 0$.

Method 1 ($\alpha = 0$ and $\beta = 1$).

$$\begin{aligned} F(x) &= \int^x e^{-\alpha s} f(s)\,ds = \int^x e^{-0s} s\,ds = \frac{x^2}{2} \\ y &= e^{\beta x}\int e^{-\beta x}[e^{\alpha x}(F(x)+C)]\,dx \\ y &= e^{x}\int e^{-x}\left[\frac{x^2}{2}+C\right]dx. \end{aligned}$$

By integration by parts one would be able to obtain

$$\int e^{-x}\left[\frac{x^2}{2}\right]dx = \cdots = -e^{-x}\left[\frac{x^2}{2}+x+1\right]+B,$$

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where B is an arbitrary constant. Hence the general solution of the given differential equation is:

$$
\begin{aligned}
y &= e^x\left\{-e^{-x}\left[\left(\frac{x^2}{2}+x+1\right)\right]+B\right\}+e^x\int Ce^{-x}dx \\
y &= -\left(\frac{x^2}{2}+x+1\right)+Be^x+e^x(-Ce^{-x}) \\
y &= A+Be^x-\left(\frac{x^2}{2}+x\right) \text{ where } A=-C-1.
\end{aligned}
$$

Method 2 ($\alpha = 1$ and $\beta = 0$).

$$
\begin{aligned}
F(x) &= \int^x e^{-\alpha s}f(s)\,ds = \int^x e^{-s}s\,ds = \cdots = -e^{-x}(x+1) \\
y &= e^{\beta x}\int e^{-\beta x}[e^{\alpha x}(F(x)+C)]dx \\
y &= e^{0x}\int e^{-0x}[e^x(-e^{-x}(x+1)+C)]dx \\
y &= \int -x-1+Ce^x dx \\
y &= A+Be^x-\left(\frac{x^2}{2}+x\right) \text{ where } C=B.
\end{aligned}
$$

Method 3 (the traditional method taught in school)

Step 1 Consider the auxiliary equation $m^2 - m = 0$ i.e. $m = 1$ or $m = 0$.

Step 2 Hence the complementary function is $y = Ae^{0x} + Be^{1x} = A + Be^x$.

Step 3 Since the right hand side function is $x = xe^{0x}$ i.e. repeating one of the complementary functions e^{0x}, so the PI for this case is $y = x(\alpha + \beta x)$.

Step 4 Differentiate the PI and determine the values of $\alpha(=-1)$ and $\beta(=-\frac{1}{2})$ by substituting the relevant equations into the original differential equations and use comparing of coefficients.

Step 5 General solution is obtained by adding up the complementary function and PI.

However, many students do not understand the followings:

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Step 1 Why is the differential equation (with right hand side treated as zero) equivalent to the auxiliary equation? There is a need to show the bijective property i.e. m represents the 1st derivative and m^2 the 2nd derivative.

Step 2 How to relate the roots of the auxiliary equation to the complementary function? Many teachers do the reverse by using $y = Ae^{\alpha x} + Be^{\beta x}$ and through differentiation show that the 2nd order differential equation is

$$\frac{d^2y}{dx^2} - (\alpha+\beta)\frac{dy}{dx} + \alpha\beta y = 0.$$

The next question (with obvious answer) is why complementary function called complementary and not complimentary?

Step 3 Why the need to multiply the general form of the right hand side function by x or x^2 depending on how many times the right hand side function repeats the function in the complementary function? Also if the right hand side expression contains $\sin kx$ or $\cos kx$, then the PI will be $\sin kx + \cos kx$. The above method (of using integrating factor) can help illustrate this point.

Step 4 Why can we use the method of comparing coefficients to determine the unknowns in the PI? Though the reason is simple, expressions on both sides of the equation are equivalent, but why is that so?

Step 5 Why the general solution is obtained by adding up the complementary function and the PI? This can be explained in vector space when we bring in the concept of kernel (in differential equation we call them as complementary functions) and particular solution (in differential equation we call them as PI). Then this will explain why expressions on both sides of the equation are equivalent.

Though in solving a 2nd order linear equation, the traditional method is neater but it is procedural. There are many questions to be answered before we can expect students to understand why this method works. As a matter of fact, vector space is itself not an area of mathematics students would appreciate, so we might only expect students to appreciate the procedure in step 5.

If the 2nd order linear differential equation $\frac{d^2y}{dx^2} - (\alpha+\beta)\frac{dy}{dx} + \alpha\beta y = f(x)$ satisfies the condition $(\alpha+\beta)^2 < 4\alpha\beta$, then we need to employ the idea of $e^{i\theta} = \cos\theta + i\sin\theta$. Let's look at the following example.

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e.g. Find the general solution of $\frac{d^2y}{dx^2} + y = 1$.

Observe that $\alpha = i$, $\beta = -i$.

$$F(x) = \int^x e^{-\alpha s} f(s)\, ds = \int^x e^{-is} \cdot 1 ds = \frac{e^{-ix}}{-i} = e^{i(\frac{\pi}{2}-x)}.$$

$$\begin{aligned} y &= e^{-ix} \int e^{ix} \left[e^{ix} \left(e^{i(\frac{\pi}{2}-x)} + C \right) \right] dx \\ y &= e^{-ix} \int e^{i(\frac{\pi}{2}+x)} + Ce^{i(2x)} dx. \end{aligned}$$

Hence the general solution of the given differential equation is:

$$\begin{aligned} y &= e^{-ix} \left[\frac{e^{i(x+\frac{\pi}{2})}}{i} + C\frac{e^{i2x}}{2i} + D \right] \\ y &= \frac{e^{i\frac{\pi}{2}}}{i} + \frac{C}{2i} e^{ix} + De^{-ix}. \end{aligned}$$

By applying the idea $e^{i\theta} = \cos\theta + i\sin\theta$, we therefore obtain the general solution as:

$$y = 1 + A\sin x + B\cos x \text{ where } A = \frac{C}{2} - Di \text{ and } B = \frac{C}{2i} + D.$$

Can this method like the traditional method be used for higher order linear differential equations? Yes, so long as the roots of the auxiliary equation can be determined, the integrating factor can be applied n times if it is an nth order linear differential equation.

The method involving integrating factor may enable one to appreciate better than the traditional method but the obvious disadvantage is that it is often more tedious. By sharing my little findings, I hope more people will also share theirs and hence help our future generations to understand or at least appreciate the mathematics they learn.